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# Error analysis for bivariate fractal interpolation functions generated by 3-D perturbed iterated function systems☆

Hong-Yong Wang [a,*], Shou-Zhi Yang [b], Xiu-Juan Li [c]

[a] Department of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210003, PR China

[b] Department of Mathematics, Shantou University, Shantou 515063, PR China

[c] College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China

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ABSTRACT

Based on a determined 3-D iterated function system (IFS), we introduce a perturbed IFS in $\mathbb{R}^3$. The attractor of the perturbed IFS is the graph of a bivariate fractal interpolation function (FIF) that interpolates arbitrarily given data on rectangular grids of $\mathbb{R}^2$. We consider the error problem between the FIF generated by the perturbed IFS and the FIF generated by the original IFS. An explicit relation of the difference between the two bivariate FIFs is presented. Furthermore, we investigate the error of moment integrals of the two FIFs. An upper bound estimate for the error of moments is obtained.



## 1. Introduction

Based on the theory of iterated function systems (IFSs), Barnsley introduced a class of fractal interpolation functions (FIFs) in [1,2]. These univariate continuous functions defined on closed intervals in $\mathbb{R}$ are generated by 2-D IFSs, the graphs of which interpolate a given set of data in $\mathbb{R}^2$.

In recent years, many researchers have generalized the notion of FIFs from different aspects. One of the generalizations is the construction of multivariate FIFs based on higher-dimensional IFSs. Massopust [3,4] was the first who considered bivariate FIFs on triangular regions in the special case where the interpolation points on the boundary of the domain are coplanar. In view of the lack of flexibility for this construction, Geronimo and Hardin [5] generalized this construction to allow the use of arbitrary interpolation points. In [6], Hardin and Massopust investigated $\mathbb{R}^m$-valued multivariate FIFs. Xie and Sun in [7,8] used 3-D IFSs to produce compact sets in $\mathbb{R}^3$ that contain the interpolation points defined on a rectangular region. The resulting compact sets were applied to model the rough surfaces of rock fractures. Dalla [9] improved the construction in [7,8] and gave some conditions so that the IFS can generate a continuous bivariate fractal interpolation surface (FIS). Malysz [10] studied a special construction of bivariate FIS, and gave the exact values for the Minkowski dimension of the bivariate FIS. Wang [11] considered a more general case of IFS in $\mathbb{R}^3$ and constructed a wide class of bivariate FISs defined on rectangular regions. Bouboulis, Dalla and Drakopoulos [12] made use of recurrent IFSs to yield a more flexible class of FISs suitable to approximate any natural surface. They used their methods to reconstruct complicated images and gained more satisfactory results than other fractal techniques. However, these constructions mentioned above still have some difficulties

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* Corresponding author.
*E-mail addresses:* 98wanghy@163.com (H.-Y. Wang), szyang@stu.edu.cn (S.-Z. Yang).

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that have not been overcome yet. Most of them used either interpolation points that are restricted to be collinear on the boundary of the domain or the contractive maps with the same vertical scaling factors. Recently, Bouboulis and Dalla [13] presented a new method of construction of FIS to try to solve these problems. They used FIF to construct FIS that interpolates arbitrary set of data on the grids of a rectangle. In addition, they in [14] extended the construction of FIFs to allow arbitrary interpolation points on rectangular grids of $\mathbb{R}^n$. In the same paper the fractal dimensions of a class of FIFs were derived and the methods of the construction of functions of class $C^p$ using recurrent IFSs were also presented.

At present, the theory of fractal interpolation has been widely applied to metallurgy, physics, chemistry, image processing, computer graphics and other fields needed to construct complicated objects, and has become a powerful tool in applied science and engineering. In computer graphics, the graphs of FIFs, as the attractors of IFSs, are usually used to approximate natural scenes. A considerable problem is how the corresponding FIFs will change when the IFSs generating the attractors are slightly perturbed. It is very important for the fractal approximation and reconstruction since the variations of attractors will influence the effects of approximation and reconstruction directly. In [15] this problem was examined in the case of the one variable FIFs. The authors proved that the error of the corresponding univariate FIF will also be very small provided that the IFS has a small perturbation. A similar problem concerning the perturbation of interpolation points as to how to influence the values of the corresponding FIFs was addressed in [13,16] in the case of the univariate FIFs.

In this paper, we will investigate the perturbation error problem of bivariate FIFs, which is caused by 3-D perturbed IFSs. Based on a determined IFS, which is a special case of a recurrent IFS, we first introduce a 3-D perturbed IFS in $\mathbb{R}^3$. Then we analyze the difference between the two bivariate FIFs generated by the perturbed and original IFSs, respectively. An explicit expression for the difference will be presented. Finally, we study the moment error of the two FIFs mentioned above, and give an upper bound estimate for the error.

## 2. A class of bivariate FIFs and their perturbation similitude

Let $I = [a, b]$, $J = [c, d]$ and $D = I \times J$. It is given any partition of $I$ and $J$ by $a = x_0 < x_1 < \cdots < x_M = b$ and $c = y_0 < y_1 < \cdots < y_N = d$, respectively, where $M > 1$, $N > 1$ are two determined positive integers. Set $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$, and $D_{ij} = I_i \times J_j$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$. We obtain a mesh partition of $D$, i.e. $\{D_{ij} : i = 1, 2, \ldots, M, j = 1, 2, \ldots, N\}$. Let $\Delta = \{(x_i, y_j, z_{ij}) \in D \times \mathbb{R} : i = 0, 1, \ldots, M; j = 0, 1, \ldots, N\}$ be a set of data (interpolation points) on the grids. Furthermore, let $\hat{\Delta} = \{(\hat{x}_k, \hat{y}_l, \hat{z}_{kl}) \in D \times \mathbb{R} : k = 0, 1, \ldots, K; l = 0, 1, \ldots, L\}$ be a subset of $\Delta$ such that $a = \hat{x}_0 < \hat{x}_1 < \cdots < \hat{x}_K = b$ and $c = \hat{y}_0 < \hat{y}_1 < \cdots < \hat{y}_L = d$. Consequently, the points of $\hat{\Delta}$ divide $D$ into $K \cdot L$ rectangles $\hat{D}_{kl} = [\hat{x}_{k-1}, \hat{x}_k] \times [\hat{y}_{l-1}, \hat{y}_l]$, $k = 1, 2, \ldots, K$ and $l = 1, 2, \ldots, L$. Let $\mathbb{J} : \{1, 2, \ldots, M\} \times \{1, 2, \ldots, N\} \to \{1, 2, \ldots, K\} \times \{1, 2, \ldots, L\}$ be a labelling map with $\mathbb{J}(i, j) = (k, l)$ such that $\hat{x}_k - \hat{x}_{k-1} > x_i - x_{i-1}$ and $\hat{y}_l - \hat{y}_{l-1} > y_j - y_{j-1}$ for $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$.

Define $M \cdot N$ mappings $F_{ij} : D \times \mathbb{R} \to D \times \mathbb{R}$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, by

$$F_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_i(x) \\ v_j(y) \\ w_{ij}(x, y, z) \end{pmatrix} = \begin{pmatrix} a_i x + b_i \\ c_j y + d_j \\ s_{ij} z + \varphi_{ij}(x, y) \end{pmatrix}, \tag{1}$$

for all $(x, y, z) \in D \times \mathbb{R}$, where $a_i$, $b_i$, $c_j$ and $d_j$ are some real constants, the parameters $s_{ij}$ are called vertical scaling factors with $0 < |s_{ij}| < 1$, the $\varphi_{ij} : D \to \mathbb{R}$ are bivariate Lipschitz functions. We confine each map $F_{ij}$ so that it maps the interpolation points lying on the vertices of $\hat{D}_{kl} = \hat{D}_{\mathbb{J}(i,j)}$ to the interpolation points lying on the vertices of $D_{ij}$. Hence, we have

$$\begin{cases} F_{ij}(\hat{x}_{k-1}, \hat{y}_{l-1}, \hat{z}_{k-1,l-1})^{\mathrm{T}} = (x_{i-1}, y_{j-1}, z_{i-1,j-1})^{\mathrm{T}}; \\ F_{ij}(\hat{x}_k, \hat{y}_{l-1}, \hat{z}_{k,l-1})^{\mathrm{T}} = (x_i, y_{j-1}, z_{i,j-1})^{\mathrm{T}}; \\ F_{ij}(\hat{x}_{k-1}, \hat{y}_l, \hat{z}_{k-1,l})^{\mathrm{T}} = (x_{i-1}, y_j, z_{i-1,j})^{\mathrm{T}}; \\ F_{ij}(\hat{x}_k, \hat{y}_l, \hat{z}_{kl})^{\mathrm{T}} = (x_i, y_j, z_{ij})^{\mathrm{T}}, \end{cases} \tag{2}$$

where $A^{\mathrm{T}}$ denotes the transpose matrix of $A$.

It is easy to prove that there exists a metric, which is equivalent to the Euclidean metric, such that each $F_{ij}$ is strictly contractive in this metric.

Let $\Phi : \{1, 2, \ldots, M\} \times \{1, 2, \ldots, N\} \to \{1, 2, \ldots, M \cdot N\}$ be a 1–1 function (an enumeration of the set $\{(i, j) : i = 1, 2, \ldots, M; j = 1, 2, \ldots, N\}$) such that $\Phi(i, j) = i + (j - 1)M$, $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. Define an $M \cdot N \times M \cdot N$ row-stochastic matrix $P = (p_{nm})$ as follows:

$$p_{nm} = \begin{cases} \gamma_{nm}, & \text{if } D_{\Phi^{-1}(n)} \subseteq \hat{D}_{\mathbb{J}(\Phi^{-1}(m))}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\gamma_{nm}$ are some positive numbers such that $\sum_{m=1}^{M \cdot N} p_{nm} = 1$, for $n = 1, 2, \ldots, M \cdot N$. For example, one may take $\gamma_{nm}$ to be $\frac{1}{q_n}$, where $q_n$ denotes the number of non-zero elements of the $n$th row of $(p_{nm})$ (see [12,14]). The number $p_{nm}$ gives the probability of transfer from the state $n$ into the state $m$ for a certain discrete time Markov process, which was described somewhat in detail in [12,14,17].

With the above preparations, we get a recurrent IFS $\{D \times \mathbb{R}, F_{ij}, P : i = 1, 2, \ldots, M; j = 1, 2, \ldots, N\}$ (or briefly $\{D \times \mathbb{R}, F_{1-M,1-N}, P\}$), which consists of the IFS $\{D \times \mathbb{R}, F_{1-M,1-N}\}$ associated with the set of data $\Delta$, together with an irreducible row-stochastic matrix $P$. The recurrent structure is given by the $M \cdot N \times M \cdot N$ connection matrix $C = (C_{nm})$, which is defined by $C_{nm} = 1$ if $p_{mn} > 0$ and $C_{nm} = 0$ otherwise.

It has been proved in [12,14] that the recurrent IFS $\{D \times \mathbb{R}, F_{1-M,1-N}, P\}$ has a unique attractor $G$. In general, $G$ is a compact subset of $\mathbb{R}^3$ containing the points of $\Delta$. Bouboulis and Dalla [14] gave some conditions so that $G$ is the graph of a continuous function $f$.

**Proposition 1** *([14]). Let $h \in C(D)$ be a Lipschitz function that interpolates the points of $\Delta$. If the recurrent IFS defined above satisfies the conditions*

$$\begin{cases} w_{ij}(\hat{x}_k, y, h(\hat{x}_k, y)) = h(x_i, v_j(y)), \\ w_{ij}(\hat{x}_{k-1}, y, h(\hat{x}_{k-1}, y)) = h(x_{i-1}, v_j(y)), \\ w_{ij}(x, \hat{y}_l, h(x, \hat{y}_l)) = h(u_i(x), y_j), \\ w_{ij}(x, \hat{y}_{l-1}, h(x, \hat{y}_{l-1})) = h(u_i(x), y_{j-1}), \end{cases} \tag{3}$$

*where $(k, l) = \mathbb{J}(i, j)$, for all $(x, y) \in \hat{D}_{\mathbb{J}(i,j)}$, $(i, j) \in \{1, 2, \dots, M\} \times \{1, 2, \dots, N\}$, then its attractor $G$ is the graph of a continuous function $f$ that interpolates the data set $\Delta$. Moreover, $f$ satisfies the functional equation*

$$f(x, y) = w_{ij}(u_i^{-1}(x), v_j^{-1}(y), f(u_i^{-1}(x), v_j^{-1}(y))) \tag{4}$$

*for all $(x, y) \in D_{ij}$, $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$.*

We refer to the function $f$ appearing in the above proposition as a recurrent FIF. Clearly, the construction of $f$ depends on the selection of $h$. Noting that Proposition 1 involves only the points of the boundary $\partial D_{ij}$ of $D_{ij}$, $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$, therefore we may take $h$ to be the Lipschitz function interpolating the points of $\Delta$ defined on $\bigcup_{i,j} \partial D_{ij}$. One possible and simple selection of $h$ is, for example, as the bilinear interpolation function whose graph on each $\partial D_{ij}$ is the spatial quadrilateral, which consists of the closed polygonal line connecting the four interpolation points of $\Delta$.

If all elements of the corresponding connection matrix $C$ are equal to one for the recurrent IFS $\{D \times \mathbb{R}, F_{1-M,1-N}, P\}$, then the resulting bivariate recurrent FIF becomes an ordinary bivariate FIF. In this case, all $\hat{D}_{\mathbb{J}(i,j)}$, $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$, are the same as $D$. Henceforth, we only consider the perturbation properties of this class of bivariate FIFs and discuss the error estimates of their moments.

For explicitness, we presume that a positive irreducible row-stochastic matrix $P = (p_{nm})$ has been given and the function $h$ in Proposition 1 has also been chosen.

From (1)–(4), we write specifically the functions $u_i : I \to I_i$, $i = 1, 2, \dots, M$ and $v_j : J \to J_j$, $j = 1, 2, \dots, N$, as

$$\begin{cases} u_i(x) = a_i(x - x_0) + x_{i-1}, \\ v_j(y) = c_j(y - y_0) + y_{j-1}, \end{cases} \quad i = 1, 2, \dots, M, j = 1, 2, \dots, N, \tag{5}$$

where $a_i = \frac{x_i - x_{i-1}}{x_M - x_0} > 0$, $c_j = \frac{y_j - y_{j-1}}{y_N - y_0} > 0$. The functions $w_{ij} : D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$, satisfy the conditions

$$\begin{cases} w_{ij}(x_0, y_0, z_{00}) = z_{i-1,j-1}; \quad w_{ij}(x_M, y_0, z_{M0}) = z_{i,j-1}; \\ w_{ij}(x_0, y_N, z_{0N}) = z_{i-1,j}; \quad w_{ij}(x_M, y_N, z_{MN}) = z_{ij}, \end{cases} \tag{6}$$

while the Lipschitz functions $\varphi_{ij} : D \to \mathbb{R}$ obey the following boundary conditions

$$\begin{cases} \varphi_{ij}(x_M, y) = h(x_i, v_j(y)) - s_{ij} h(x_M, y), \\ \varphi_{ij}(x_0, y) = h(x_{i-1}, v_j(y)) - s_{ij} h(x_0, y), \\ \varphi_{ij}(x, y_N) = h(u_i(x), y_j) - s_{ij} h(x, y_N), \\ \varphi_{ij}(x, y_0) = h(u_i(x), y_{j-1}) - s_{ij} h(x, y_0), \end{cases} \tag{7}$$

for $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$, and $\forall (x, y) \in D$. Consequently,

$$\{D \times \mathbb{R}, F_{ij} : i = 1, 2, \dots, M; j = 1, 2, \dots, N\} \tag{8}$$

constitutes a determined IFS in $\mathbb{R}^3$, which is associated with the given matrix $P$ and the chosen function $h$, and its attractor $G$ is the graph of a bivariate continuous function $f$ interpolating the arbitrarily given data set $\Delta$. In addition, $f$ satisfies the fixed point equation

$$f(x, y) = s_{ij} f(u_i^{-1}(x), v_j^{-1}(y)) + \varphi_{ij}(u_i^{-1}(x), v_j^{-1}(y)), \quad \forall (x, y) \in D_{ij}, \tag{9}$$

for $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$.

Based on the IFS (8), we now construct an IFS with perturbation terms. Define functions $T_{ij} : D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, M$, $j = 1, 2, \dots, N$, by

$$T_{ij}(x, y, z) = s_{ij} z + \varphi_{ij}(x, y) + \varepsilon_{ij} \lambda_{ij}(x, y),$$

where $\varepsilon_{ij}$ are called the perturbation parameters obeying $0 < |\varepsilon_{ij}| < 1$, $\lambda_{ij}(x, y)$ are bivariate Lipschitz functions defined on $D$, and with conditions

$$\lambda_{ij}(x_0, y) = \lambda_{ij}(x_M, y) = \lambda_{ij}(x, y_0) = \lambda_{ij}(x, y_N) = 0, \quad \forall (x, y) \in D. \tag{10}$$

Obviously, there are many bivariate Lipschitz functions $\lambda_{ij}$ such that the conditions (10) hold, for example, let $\lambda_{ij}(x, y) = (x - x_0)(y - y_0)(x_M - x)(y_N - y)\mu_{ij}(x, y)$, where $\mu_{ij}(x, y), i = 1, 2, \dots, M, j = 1, 2, \dots, N$, may take distinct differentiable functions defined on $D$. So, we get an IFS

$$\{D \times \mathbb{R}, (u_i(x), v_j(y), T_{ij}(x, y, z)) : i = 1, 2, \dots, M; j = 1, 2, \dots, N\}, \tag{11}$$

which is associated with the same matrix $P$ and function $h$ as those of the IFS (8), and is called a perturbed IFS of the IFS (8).

Let $\phi_{ij}(x, y) = \varphi_{ij}(x, y) + \varepsilon_{ij}\lambda_{ij}(x, y)$. It is easy to verify that $T_{ij}(x, y, z)$ obey the conditions (6) and $\phi_{ij}(x, y)$ are bivariate Lipschitz functions satisfying the boundary conditions (7). Hence, the IFS (11) generates a bivariate FIF, denoted by $f_\varepsilon(x, y)$, whose graph is the FIS passing through the set $\Delta$ of data.

## 3. The perturbation error analysis for bivariate FIFs

In this section we will consider the error problem between the FIFs $f_\varepsilon(x, y)$ and $f(x, y)$ given in the previous section. In order to get an explicit expression for the difference between $f_\varepsilon(x, y)$ and $f(x, y)$, we first present a useful lemma which gives a multiresolution representation of $f(x, y)$.

To simplify our notations, we set $\mathcal{M} = \{1, 2, \dots, M\}$, and $\mathcal{N} = \{1, 2, \dots, N\}$. For $\forall (x, y) \in D$, let $u_{i_1 i_2 \cdots i_n}(x) = u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_n}(x)$, $i_k \in \mathcal{M}, k = 1, 2, \dots, n$, and let $v_{j_1 j_2 \cdots j_n}(y) = v_{j_1} \circ v_{j_2} \circ \cdots \circ v_{j_n}(y), j_k \in \mathcal{N}, k = 1, 2, \dots, n$. Using the successive iteration and inductive method, we can deduce the following lemma from (5) and (9).

**Lemma 1.** *Let $f(x, y)$ be the FIF generated by IFS (8). Then for any $(x, y) \in D$, $i_k \in \mathcal{M}, j_k \in \mathcal{N}, k = 1, 2, \dots, n$, we have*

$$u_{i_1 i_2 \cdots i_n}(x) = \left(\prod_{k=1}^{n} a_{i_k}\right)(x - x_0) + \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r} a_{i_k}\right)(x_{i_{r+1}-1} - x_0) + x_{i_1 - 1}, \tag{12}$$

$$v_{j_1 j_2 \cdots j_n}(y) = \left(\prod_{k=1}^{n} c_{j_k}\right)(y - y_0) + \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r} c_{j_k}\right)(y_{j_{r+1}-1} - y_0) + y_{j_1 - 1}, \tag{13}$$

$$f(u_{i_1 i_2 \cdots i_n}(x), v_{j_1 j_2 \cdots j_n}(y)) = \left(\prod_{k=1}^{n} s_{i_k j_k}\right) f(x, y) + \left(\prod_{k=1}^{n-1} s_{i_k j_k}\right)\varphi_{i_n j_n}(x, y) + \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} s_{i_k j_k}\right)\varphi_{i_r j_r}(u_{i_{r+1}\cdots i_n}(x), v_{j_{r+1}\cdots j_n}(y)). \tag{14}$$

**Remark 1.** In the calculation of $\prod_{k=1}^{n-1} s_{i_k j_k}$, we set $\prod_{k=1}^{0} s_{i_k j_k} = 1$ when $n = 1$.

For $1 \le r \le n - 1$, similarly to (12) and (13), we have

$$u_{i_{r+1}\cdots i_n}(x) = \left(\prod_{k=1}^{n-r} a_{i_{r+k}}\right)(x - x_0) + \sum_{l=1}^{n-r-1}\left(\prod_{k=1}^{l} a_{i_{r+k}}\right)(x_{i_{r+1+l}-1} - x_0) + x_{i_{r+1}-1}, \tag{15}$$

$$v_{j_{r+1}\cdots j_n}(y) = \left(\prod_{k=1}^{n-r} c_{j_{r+k}}\right)(y - y_0) + \sum_{l=1}^{n-r-1}\left(\prod_{k=1}^{l} c_{j_{r+k}}\right)(y_{j_{r+1+l}-1} - y_0) + y_{j_{r+1}-1}. \tag{16}$$

We now present the following Theorem 1 by means of Lemma 1.

**Theorem 1.** *Let $f(x, y)$ and $f_\varepsilon(x, y)$ be the bivariate FIFs generated by the IFSs (8) and (11), respectively. For any $(x, y) \in D$, let $\{i_k\}, i_k \in \mathcal{M}$, be the sequence such that $x$ satisfies*

$$x = \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r} a_{i_k}\right)(x_{i_{r+1}-1} - x_0) + x_{i_1 - 1}, \tag{17}$$

*and let $\{j_k\}, j_k \in \mathcal{N}$, be another sequence such that $y$ satisfies*

$$y = \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r} c_{j_k}\right)(y_{j_{r+1}-1} - y_0) + y_{j_1 - 1}. \tag{18}$$

*Then*

$$\begin{aligned} f_\varepsilon(x, y) - f(x, y) = \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1} s_{i_k j_k}\right)\varepsilon_{i_r j_r}\lambda_{i_r j_r}&\left(\sum_{l=1}^{\infty}\left(\prod_{k=1}^{l} a_{i_{r+k}}\right)(x_{i_{r+1+l}-1} - x_0) + x_{i_{r+1}-1},\right.\\ &\left.\sum_{l=1}^{\infty}\left(\prod_{k=1}^{l} c_{j_{r+k}}\right)(y_{j_{r+1+l}-1} - y_0) + y_{j_{r+1}-1}\right). \end{aligned} \tag{19}$$

**Proof.** Let $u_{i_1 i_2 \ldots i_n}(I) = u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_n}(I)$, $i_k \in \mathcal{M}$, $k = 1, 2, \ldots, n$. Since each $u_{i_k}(x)$ is contractive on the closed interval $I$, the sequence of sets $\{u_{i_1 i_2 \cdots i_n}(I)\}$ is monotone decreasing. Hence, $\bigcap_{n=1}^{\infty} u_{i_1 i_2 \cdots i_n}(I)$ consists of a single point in $I$ for any sequence $\{i_k\}$, $i_k \in \mathcal{M}$. Obviously, for any given $x \in I$, there exists a sequence $\{i_k\}$, $i_k \in \mathcal{M}$, such that

$$\{x\} = \bigcap_{n=1}^{\infty} u_{i_1 i_2 \cdots i_n}(I) = \lim_{n \to \infty} u_{i_1 i_2 \cdots i_n}(I).$$

Noticing that each $a_{i_k}$ in (12) obeys $0 < a_{i_k} < 1$, thus, applying (12), we can express $x$ as

$$x = \lim_{n \to \infty} u_{i_1 i_2 \cdots i_n}(\bar{x}) = \sum_{r=1}^{\infty} \left( \prod_{k=1}^{r} a_{i_k} \right) (x_{i_{r+1}-1} - x_0) + x_{i_1 - 1},$$

where $\bar{x}$ is chosen arbitrarily in $I$. Similarly, for any given $y \in J$, there exists another sequence $\{j_k\}, j_k \in \mathcal{N}$, such that (18) holds.

For any $(x, y) \in D$, using (14)–(16), we can express $f(x, y)$ as

$$\begin{aligned} f(x,y) &= \lim_{n \to \infty} f(u_{i_1 i_2 \cdots i_n}(\bar{x}), v_{j_1 j_2 \cdots j_n}(\bar{y})) \\ &= \sum_{r=1}^{\infty} \left( \prod_{k=1}^{r-1} s_{i_k j_k} \right) \varphi_{i_r j_r} \left( \sum_{l=1}^{\infty} \left( \prod_{k=1}^{l} a_{i_{r+k}} \right) (x_{i_{r+1+l}-1} - x_0) + x_{i_{r+1}-1}, \sum_{l=1}^{\infty} \left( \prod_{k=1}^{l} c_{j_{r+k}} \right) (y_{j_{r+1+l}-1} - y_0) + y_{j_{r+1}-1} \right), \end{aligned} \tag{20}$$

where $(\bar{x}, \bar{y})$ may be chosen arbitrarily in $D$. Set

$$\begin{cases} x' = \sum_{l=1}^{\infty} \left( \prod_{k=1}^{l} a_{i_{r+k}} \right) (x_{i_{r+1+l}-1} - x_0) + x_{i_{r+1}-1}, \\ y' = \sum_{l=1}^{\infty} \left( \prod_{k=1}^{l} c_{j_{r+k}} \right) (y_{j_{r+1+l}-1} - y_0) + y_{j_{r+1}-1}. \end{cases} \tag{21}$$

It is easy to see from (15) and (16) that $(x', y')$ belongs to $D$.

Also, a similar argument yields

$$f_\varepsilon(x,y) = \sum_{r=1}^{\infty} \left( \prod_{k=1}^{r-1} s_{i_k j_k} \right) \left[ \varphi_{i_r j_r}(x', y') + \varepsilon_{i_r j_r} \lambda_{i_r j_r}(x', y') \right]. \tag{22}$$

Thus, (19) follows from (20) and (22). $\square$

From Theorem 1, we may easily deduce the following corollary, which provides an upper bound for the difference between $f_\varepsilon(x, y)$ and $f(x, y)$.

**Corollary 1.** *Let $f(x, y)$ and $f_\varepsilon(x, y)$ be the bivariate FIFs generated by the IFSs (8) and (11), respectively. Let $s = \max\{|s_{ij}| : i \in \mathcal{M}, j \in \mathcal{N}\} < 1$, $\varepsilon = \max\{|\varepsilon_{ij}| : i \in \mathcal{M}, j \in \mathcal{N}\} < 1$, and $A = \max\{\| \lambda_{ij} \|_\infty : i \in \mathcal{M}, j \in \mathcal{N}\}$, where $\| \lambda_{ij} \|_\infty = \max\{|\lambda_{ij}(x, y)| : (x, y) \in D\}$. Then*

$$|f_\varepsilon(x,y) - f(x,y)| \le \frac{A}{1-s} \varepsilon, \quad \forall (x,y) \in D. \tag{23}$$

**Remark 2.** From (23), we can see that the upper bound for the error between $f_\varepsilon(x, y)$ and $f(x, y)$ is $O(\varepsilon)$. It means that the perturbation error of the corresponding FIF will be very small provided that a slight perturbation occurs in the IFS. From the viewpoint of fractal surface reconstruction, the overall shape of the resulting FIS will not change violently when the third component of the original IFS in $\mathbb{R}^3$ undergoes a small perturbation.

We give a simple example to illustrate the influence of the perturbation terms on the corresponding FIS.

**Example.** Let $D = [0, 10] \times [0, 10]$. Given a set $\Delta$ of $3 \times 3$ data points $(x_i, y_j, z_{ij})$, $i, j = 0, 1, 2$, by

$$\Delta = \{(0, 0, 4); (0, 5, 7); (0, 10, 10); (5, 0, 6); (5, 5, 14); (5, 10, 8); (10, 0, 8); (10, 5, 7); (10, 10, 6)\}.$$

We choose $h$ as the bilinear interpolant defined on $\bigcup_{i,j=1}^{2} \partial D_{ij}$, i.e. the graph of $h$ on each $\partial D_{ij}$ is the spatial quadrilateral, and take the probabilities $p_{nm} = 0.25$ for all $n, m = 1, 2, 3, 4$. Let $s_{ij} = 0.6$ for all $i, j = 1, 2$. Assume that $\varphi_{ij}(x, y) = e_{ij}x + f_{ij}y + q_{ij}xy + k_{ij}$, and its coefficients $e_{ij}, f_{ij}, q_{ij}$ and $k_{ij}$ are uniquely determined by the conditions (6) and the given $s_{ij}$. Then it is easy to check that $\varphi_{ij}$ satisfy the conditions (7). So, an IFS consisting of four mappings is defined. Choose all $\lambda_{ij} = xy(10 - x)(10 - y)$ in the corresponding perturbation IFS. Fig. 1 shows the FIS generated by the original IFS, which passes through the given set $\Delta$, and Figs. 2 and 3 show the FISs generated by the perturbed IFSs.

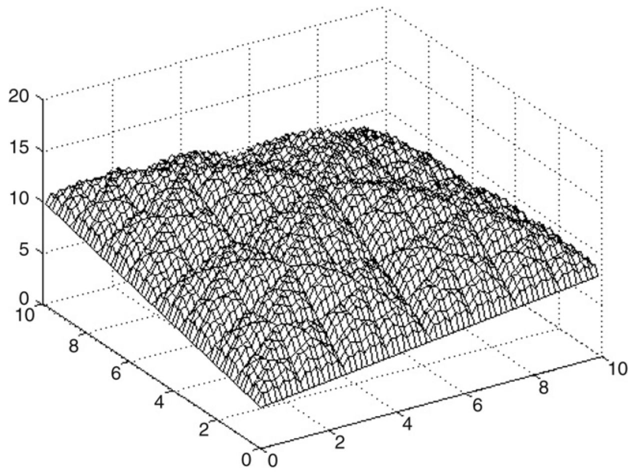


**Fig. 1.** FIS generated by the original IFS.

## 4. Moment relations for bivariate FIFs

Many researchers have shown (see, e.g. [18–20]) that the moments of fractal functions have important applications to solution of fractal inverse problems. In this section, we will proceed to discuss the relations between the moment of $f_\varepsilon(x, y)$ and the moment of $f(x, y)$. Two cases of moments will be considered, one is the case in which $f(x, y)$ is defined on the whole region $D$, and the other is that in which $f(x, y)$ is regarded as a function defined on an $n$-order subregion of $D$. The error estimates for the moments are made in two cases. The definition of the moment integral of a continuous function is first given below.

**Definition.** Let $g(x, y)$ be a continuous function defined on a bounded closed region $\Omega$ in $\mathbb{R}^2$, and $p, q$ and $t$ three non-negative integers. Then the integral $\iint_\Omega x^p y^q [g(x, y)]^t \mathrm{d}x\mathrm{d}y$ is termed the $(p, q, t)$-order moment of $g(x, y)$ on $\Omega$, denoted by $M_\Omega(g; p, q, t)$.

**Theorem 2.** *Let $f(x, y)$ and $f_\varepsilon(x, y)$ be the FIFs generated by the IFSs (8) and (11), respectively. Then for arbitrary integers $p \geq 0$, $q \geq 0$, and $t \geq 1$, we have*

$$|M_D(f_\varepsilon; p, q, t) - M_D(f; p, q, t)| \leq \sigma t h_1^p h_2^q \left(\frac{A}{1-s}\right)^t (B+\varepsilon)^{t-1}\varepsilon,$$

*where $\sigma$ denotes the area of region $D = [a, b] \times [c, d]$, $h_1 = \max\{|a|, |b|\}$, $h_2 = \max\{|c|, |d|\}$, $B = (1-s)\|f\|_\infty / A$, and $s$, $\varepsilon$ and $A$ are defined as in (23).*

**Proof.** For any $(x, y) \in D$, let $\{i_k\}$, $i_k \in \mathcal{M}$, and $\{j_k\}$, $j_k \in \mathcal{N}$, be two sequences such that (17) and (18) hold. By the definition of moment, we have

$$\begin{aligned} M_D(f_\varepsilon; p, q, t) &= \iint_D x^p y^q [f_\varepsilon(x, y)]^t \mathrm{d}x\mathrm{d}y = \sum_{\substack{i\in\mathcal{M}\\ j\in\mathcal{N}}} \iint_{D_{ij}} x^p y^q [f_\varepsilon(x, y)]^t \mathrm{d}x\mathrm{d}y \\ &= \sigma^{-1} \sum_{\substack{i\in\mathcal{M}\\ j\in\mathcal{N}}} \sigma_{ij} \iint_D [u_i(x)]^p [v_j(y)]^q [f_\varepsilon(u_i(x), v_j(y))]^t \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{24}$$

where $\sigma_{ij}$ denotes the area of subregion $D_{ij}$. Applying the fixed point Eq. (9) and the formula (19), we obtain

$$f_\varepsilon(u_i(x), v_j(y)) = s_{ij} f_\varepsilon(x, y) + \varphi_{ij}(x, y) + \varepsilon_{ij}\lambda_{ij}(x, y) = s_{ij} f(x, y) + \varphi_{ij}(x, y) + s_{ij} \sum_{r=1}^{\infty} \left(\prod_{k=1}^{r-1} s_{i_k j_k}\right) \varepsilon_{i_r j_r} \lambda_{i_r j_r}(x', y') + \varepsilon_{ij}\lambda_{ij}(x, y),$$

where $x'$ and $y'$ are defined in (21).

Let $g_{ij}(x, y) = s_{ij} \sum_{r=1}^{\infty} \left(\prod_{k=1}^{r-1} s_{i_k j_k}\right) \varepsilon_{i_r j_r} \lambda_{i_r j_r}(x', y') + \varepsilon_{ij}\lambda_{ij}(x, y)$. It is easy to verify that $g_{ij}$, $i \in \mathcal{M}, j \in \mathcal{N}$, are continuous on $D$ and satisfy $|g_{ij}(x, y)| \leq \frac{\varepsilon A}{1-s}$. Hence, from (24), we have

$$\begin{aligned} M_D(f_\varepsilon; p, q, t) &= \frac{1}{\sigma} \sum_{\substack{i\in\mathcal{M}\\ j\in\mathcal{N}}} \sigma_{ij} \iint_D [u_i(x)]^p [v_j(y)]^q [s_{ij} f(x, y) + \varphi_{ij}(x, y) + g_{ij}(x, y)]^t \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{\sigma} \sum_{\substack{i\in\mathcal{M}\\ j\in\mathcal{N}}} \sigma_{ij} \iint_D [u_i(x)]^p [v_j(y)]^q [f(u_i(x), v_j(y)) + g_{ij}(x, y)]^t \mathrm{d}x\mathrm{d}y. \end{aligned}$$

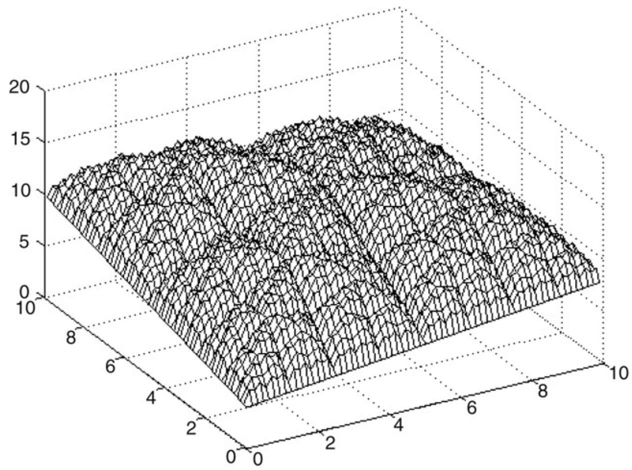


**Fig. 2.** FIS generated by the perturbed IFS with $\varepsilon_{ij} = 0.001$.

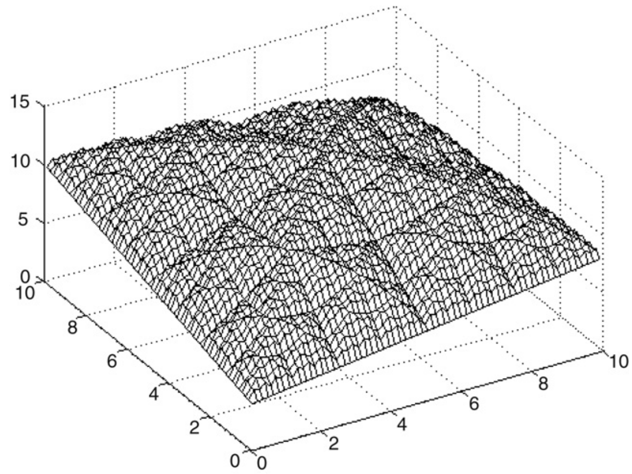


**Fig. 3.** FIS generated by the perturbed IFS with $\varepsilon_{ij} = -0.001$.

Applying the binomial formula, we obtain

$$\begin{aligned}
M_D(f_\varepsilon; p, q, t) &= \frac{1}{\sigma} \sum_{\substack{i \in \mathcal{M} \\ j \in \mathcal{N}}} \sigma_{ij} \iint_D [u_i(x)]^p [v_j(y)]^q [f(u_i(x), v_j(y))]^t \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{\sigma} \sum_{\substack{i \in \mathcal{M} \\ j \in \mathcal{N}}} \sigma_{ij} \iint_D [u_i(x)]^p [v_j(y)]^q \sum_{k=0}^{t-1} \binom{t}{k} [f(u_i(x), v_j(y))]^k [g_{ij}(x, y)]^{t-k} \mathrm{d}x\mathrm{d}y \\
&= M_D(f; p, q, t) + \frac{1}{\sigma} \sum_{\substack{i \in \mathcal{M} \\ j \in \mathcal{N}}} \sigma_{ij} \sum_{k=0}^{t-1} \binom{t}{k} \iint_D [u_i(x)]^p [v_j(y)]^q [f(u_i(x), v_j(y))]^k [g_{ij}(x, y)]^{t-k} \mathrm{d}x\mathrm{d}y. \qquad (25)
\end{aligned}$$

From the facts that $u_i(x) \in [x_{i-1}, x_i]$, and $v_j(y) \in [y_{j-1}, y_j]$ for any $(x, y) \in D$, and $i \in \mathcal{M}, j \in \mathcal{N}$, we can obtain $|u_i(x)| \le \max\{|a|, |b|\} = h_1$ and $|v_j(y)| \le \max\{|c|, |d|\} = h_2$. Thus, the absolute value of integral on the right-hand side in (25) does not exceed $\sigma h_1^p h_2^q \|f\|_\infty^k \left(\frac{\varepsilon A}{1-s}\right)^{t-k}$. Let $B = (1 - s)\|f\|_\infty / A$, then

$$\begin{aligned}
|M_D(f_\varepsilon; p, q, t) - M_D(f; p, q, t)| &\le \sum_{\substack{i \in \mathcal{M} \\ j \in \mathcal{N}}} \sigma_{ij} \sum_{k=0}^{t-1} \binom{t}{k} h_1^p h_2^q \left(\frac{A}{1-s}\right)^t B^k \varepsilon^{t-k} \\
&= \sigma h_1^p h_2^q \left(\frac{A}{1-s}\right)^t \sum_{k=0}^{t-1} \binom{t}{k} B^k \varepsilon^{t-k}
\end{aligned}$$

$$= \sigma h_1^p h_2^q \left(\frac{A}{1-s}\right)^t \varepsilon \sum_{k=0}^{t-1} \binom{t-1}{k} B^k \varepsilon^{t-1-k} \frac{t}{t-k}$$
$$\leq \sigma t h_1^p h_2^q \left(\frac{A}{1-s}\right)^t (B+\varepsilon)^{t-1} \varepsilon.$$

This completes the proof. □

Similarly to the notation $u_{i_1 i_2 \cdots i_n}(I)$, we denote $v_{j_1} \circ v_{j_2} \circ \cdots \circ v_{j_n}(J)$ by $v_{j_1 j_2 \cdots j_n}(J)$, $j_k \in \mathcal{N}$, $k = 1, 2, \ldots, n$. Let $D_{i(n)\times j(n)} = u_{i_1 i_2 \cdots i_n}(I) \times v_{j_1 j_2 \cdots j_n}(J)$. Then $D_{i(n)\times j(n)}$ is called an $n$-order subregion of $D$. In the following we will discuss the upper estimate for the difference between the moments of $f_\varepsilon$ and $f$ on $D_{i(n)\times j(n)}$.

For any $(x, y) \in D$, similarly to (14), we can get

$$f_\varepsilon(u_{i_1\cdots i_n}(x), v_{j_1\cdots j_n}(y)) = \left(\prod_{k=1}^{n} s_{i_k j_k}\right) f_\varepsilon(x, y) + \left(\prod_{k=1}^{n-1} s_{i_k j_k}\right) [\varphi_{i_n j_n}(x, y) + \varepsilon_{i_n j_n} \lambda_{i_n j_n}(x, y)]$$
$$+ \sum_{r=1}^{n-1} \left(\prod_{k=1}^{r-1} s_{i_k j_k}\right) [\varphi_{i_r j_r}(u_{i_{r+1}\cdots i_n}(x), v_{j_{r+1}\cdots j_n}(y)) + \varepsilon_{i_r j_r} \lambda_{i_r j_r}(u_{i_{r+1}\cdots i_n}(x), v_{j_{r+1}\cdots j_n}(y))]. \tag{26}$$

Hence, by (14) and (26), we have

$$f_\varepsilon(u_{i_1\cdots i_n}(x), v_{j_1\cdots j_n}(y)) - f(u_{i_1\cdots i_n}(x), v_{j_1\cdots j_n}(y)) = \left(\prod_{k=1}^{n} s_{i_k j_k}\right) [f_\varepsilon(x, y) - f(x, y)]$$
$$+ \varepsilon_{i_n j_n} \left(\prod_{k=1}^{n-1} s_{i_k j_k}\right) \lambda_{i_n j_n}(x, y) + \sum_{r=1}^{n-1} \left(\prod_{k=1}^{r-1} s_{i_k j_k}\right) \varepsilon_{i_r j_r} \lambda_{i_r j_r}(u_{i_{r+1}\cdots i_n}(x), v_{j_{r+1}\cdots j_n}(y)). \tag{27}$$

We denote by $\|I\|$ the modulus of partition of the interval $I$, i.e. $\|I\| = \max\{|x_i - x_{i-1}| : i = 1, 2, \ldots, M\}$. Similarly, the symbol $\|J\|$ denotes the modulus of partition of $J$. For any $x \in I$, and any given $(i_1, i_2, \ldots, i_n) \in \mathcal{M}^n$, we can deduce from (12) that

$$|u_{i_1\cdots i_n}(x)| \leq \left(\prod_{k=1}^{n} a_{i_k}\right)(b-a) + \sum_{r=1}^{n-1} \left(\prod_{k=1}^{r} a_{i_k}\right)(b-a) + |x_{i_1-1}|$$
$$= (b-a) \sum_{r=1}^{n} \left(\prod_{k=1}^{r} a_{i_k}\right) + |x_{i_1-1}|$$
$$\leq (b-a) \sum_{r=1}^{n} \left(\frac{\|I\|}{b-a}\right)^r + |x_{i_1-1}| \leq \frac{(b-a)\|I\|}{b-a-\|I\|} + |x_{i_1-1}|.$$

On the other hand, we have $u_{i_1\cdots i_n}(x) \in I$ since each $u_{i_k}$, $k = 1, 2, \ldots, n$, is a contractive linear mapping on $I$. Let $L_1 = \min\{\frac{(b-a)\|I\|}{b-a-\|I\|} + |x_{i_1-1}|, h_1\}$, where $h_1 = \max\{|a|, |b|\}$. Then $|u_{i_1\cdots i_n}(x)| \leq L_1$ for any $x \in I$, and $i_k \in \mathcal{M}$, $k = 1, 2, \ldots, n$. Obviously, the positive number $L_1$ depends probably on $i_1$ since $u_{i_1\cdots i_n}(x) \in u_{i_1}(I)$.

As above, we are also able to obtain that $|v_{j_1\cdots j_n}(y)| \leq L_2$ for any $y \in J$, and given $(j_1, j_2, \ldots, j_n) \in \mathcal{N}^n$, where $L_2 = \min\{\frac{(d-c)\|J\|}{d-c-\|J\|} + |y_{j_1-1}|, h_2\}$, and $h_2 = \max\{|c|, |d|\}$.

With the above preparations, by means of (27) and (19), applying the method similar to that presented in Theorem 2, we can prove the following.

**Theorem 3.** *Let $f(x, y)$ and $f_\varepsilon(x, y)$ be the FIFs generated by the IFSs (8) and (11), respectively. Then, for arbitrarily given $(i_1, i_2, \ldots, i_n) \in \mathcal{M}^n$, and $(j_1, j_2, \ldots, j_n) \in \mathcal{N}^n$, we have*

$$\left| \iint_{D_{i(n)\times j(n)}} x^p y^q [f_\varepsilon(x, y)]^t \mathrm{d}x\mathrm{d}y - \iint_{D_{i(n)\times j(n)}} x^p y^q [f(x, y)]^t \mathrm{d}x\mathrm{d}y \right| \leq \sigma^{1-n} \left(\prod_{k=1}^{n} \sigma_{i_k j_k}\right) t L_1^p L_2^q \left(\frac{A}{1-s}\right)^t (B+\varepsilon)^{t-1} \varepsilon, \tag{28}$$

*where $\sigma_{i_k j_k}$ denotes the area of subregion $D_{i_k j_k}$, and the other notations in (28) are the same as in Theorem 2.*

## 5. Conclusions

In this work, we make a sensitivity analysis for a class of bivariate FIFs which are generated by a class of 3-D IFSs. An explicit expression for the perturbation error of such FIFs is given and the upper estimates for their moment integrals are obtained in two cases of the integral domain.

Although our research proceeds on the rectangular grids of $\mathbb{R}^2$, we believe that the techniques used in this paper and the results obtained can be generalized to the case of $\mathbb{R}^n$ by means of the general construction of FIFs on grids of $\mathbb{R}^n$ presented in [14]. Furthermore, we also think that these results on perturbation error analysis for FISs and their moments will be significant in many applied areas including fractal reconstruction of rough surfaces, the modelling of 2-D signals etc. Perhaps one has already noted that the used IFS in this work is only a special type of recurrent IFSs, in which all elements for the connection matrix of the recurrent IFS are equal to one. Naturally, a question is raised: Whether or not can the techniques and results in this paper be generalized to the case of recurrent FIFs produced by the general recurrent IFSs? This is a problem that deserves consideration.

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